

On a modified scalarizing method for approximate solutions of nonconvex multi-objective optimization problems

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ABSTRACT

The theory of optimality conditions serves as a fundamental basis for finding solutions to optimization problems, both analytically and numerically. Beyond the optimality conditions for exact solutions, researchers are often interested in optimality conditions for approximate solutions. In this paper, we focus on obtaining these conditions for multi-objective optimization problems with infinitely many constraints via a modified scalarizing method.

Researchers often study approximate solutions within the feasible sets of optimization problems, such as " ϵ "-solutions or " ϵ "-quasi-solutions. Attention has also been directed toward approximate solutions that lie outside the feasible set, such as almost " ϵ "-solutions or almost " ϵ "-quasi-solutions. These concepts were initially proposed by P. Loridan for single-objective optimization problems and were later extended to the multi-objective case. Such types of solutions are well suited for numerical methods based on iterative sequences.

Recently, new concepts of " ϵ "-quasi-subdifferentials for locally Lipschitz functions and " ϵ "-quasi-normal sets have been proposed. By employing a modification of the " ϵ "-constrained scalarization method, and combining it with the new concepts above, conditions for approximate solutions of nonconvex multi-objective optimization problems with infinitely many constraints are established. These results are expressed via inclusions involving a new form of approximate subdifferentials for locally Lipschitz functions together with the new notion of approximate quasi-normal sets. Throughout this paper, we contribute alternative representations of optimality conditions to obtain approximate solutions of nonconvex multi-objective optimization problems.

Key words: ϵ -quasi subdifferentials, almost weakly ϵ -Pareto solutions, almost ϵ -Pareto solutions

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INTRODUCTION

The theory of optimality conditions serves as a foundation for solving optimization problems using both analytical and numerical methods. Researchers are also often interested in approximate optimality conditions. The study of approximate solutions to both single- and multi-objective optimization problems has attracted the attention of many researchers in recent years^{1,2,3,4,5,6,7,8,9,10}. In this paper, via a scalarization method, optimality conditions are established for approximate solutions to a multi-objective problem with infinitely many constraints. It should be noted that there are numerous scalarization methods for finding optimal solutions to multi-objective problems introduced in the literature^{11,12}.

Researchers often study approximate solutions within the feasible set of an optimization problem, such as " ϵ "-solutions or " ϵ "-quasi-solutions. Attention is also given to approximate solutions that lie outside the feasible set, such as almost " ϵ "-solutions or almost " ϵ "-quasi-solutions. These concepts were initially proposed by P. Loridan⁶ for single-objective optimization problems and were later extended to the multi-objective case. Such types of solutions can be readily obtained using iterative numerical methods. Recently, a new concept of " ϵ "-quasi-subdifferentials for locally Lipschitz functions was proposed in¹³ and then applied to establish optimality conditions related to approximate solutions of scalar problems in¹⁴. The aim of this study is to establish optimality conditions for approximate solutions of nonconvex multi-objective problems with infinite constraints. These optimality conditions will be characterized using sets of quasi-approximate subdifferentials of the relevant functions. First, we use the scalarization method for multi-objective problems via the " ϵ "-constraint method introduced by Chankong and Haimes¹¹. Through a slight modification of the structure of the scalar problem associated with the multi-objective problem, we establish relationships between the

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approximate solutions of the two problems. Accordingly, we derive approximate optimality conditions for multi-objective problems.

The structure of the remainder of the paper is as follows. Section 2 recalls basic concepts related to locally Lipschitz functions, followed by the notion of an ε -quasi-subdifferential and its calculus rules, the set of all ε -normal vectors at a point in a given closed set, and results on approximate optimality conditions of single-objective problems. Section 3 presents the main results, which establish approximate optimality conditions for scalar problems associated with multi-objective problems corresponding to the scalar method. Next, we derive the optimality conditions for approximate Pareto solutions as well as almost approximate Pareto solutions for multi-objective problems. In particular, the question of whether approximate Pareto solutions exist in multi-objective problems is also investigated, and a numerical example is provided.

PRELIMINARIES

Let g be a real-valued function on \mathbb{R}^n . The function g is called locally Lipschitz at a point $z \in \mathbb{R}^n$ if there exist a neighborhood $U(z) \subset \mathbb{R}^n$ and a number $K > 0$ such that

$$|g(x) - g(y)| \leq K\|x - y\|, \forall x, y \in U(z).$$

If this inequality holds for all $x, y \in \mathbb{R}^n$, the function g is called Lipschitz on \mathbb{R}^n .

Let Ω be an arbitrary index set (possibly infinite). For each $v \in \Omega$, let $h_v: \mathbb{R}^n \rightarrow \mathbb{R}$. The family $(h_v)_{v \in \Omega}$ is said to be locally Lipschitz in x , uniform in v , if there exist a neighborhood $U(x) \subset \mathbb{R}^n$ and a constant $K > 0$ such that for any $y, z \in U(x)$ and any $v \in \Omega$, we have

$$|h_v(y) - h_v(z)| \leq K\|y - z\|.$$

For $g: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the one-sided directional derivative is defined by

$$g'(z; d) := \lim_{t \downarrow 0} \frac{g(z + td) - g(z)}{t}.$$

The directional derivative at $z \in \text{dom}g$ in the direction $d \in \mathbb{R}^n$ and the subdifferential of the function g in the sense of Clarke⁷, respectively, are denoted and defined as follows:

$$g^c(z; d) := \limsup_{\substack{x \rightarrow z \\ t \downarrow 0}} \frac{g(x + td) - g(x)}{t},$$

$$\partial^c g(z) := \{u \in \mathbb{R}^n \mid \langle u, d \rangle \leq g^c(z; d), \forall d \in \mathbb{R}^n\},$$

where $\text{dom}g := \{x \in \mathbb{R}^n \mid |g(x)| < +\infty\}$. If $g'(z; d) = g^c(z; d)$ for every $d \in \mathbb{R}^n$, we say that the function g is regular at z ¹⁵.

A locally Lipschitz function $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is called quasi-convex if $g(y) \leq g(x) \Rightarrow g^c(x; y - x) \leq 0$, and it is called pseudoconvex if $g^c(x; y - x) \geq 0 \Rightarrow g(y) \geq g(x)$ for all $x, y \in \mathbb{R}^n$ ¹⁶.

The normal cone of a closed set $S \subset \mathbb{R}^n$ at $z \in S$ is given by

$$N(S, z) := \{u \in \mathbb{R}^n \mid \langle u, v \rangle \leq 0, \forall v \in T(S, z)\}, \tag{2.1}$$

where $T(S, z) := \{v \in \mathbb{R}^n \mid d_S^c(z; v) = 0\}$ is the tangent cone of S at $z \in S$, and d_S is the distance function from z to S ¹⁷. When S is a convex set, the cone described above is identical to the one found in convex analysis, that is,

$$N(S, z) := \{u \in \mathbb{R}^n \mid \langle u, x - z \rangle \leq 0, \forall x \in S\}.$$

Based on this notion, for $S \subset \mathbb{R}^n$ and $z \in S$, the set of all ε -normal vectors¹³ of a point $z \in S$ is given by

$$N_\varepsilon^Q(S, z) := \{u \in \mathbb{R}^n \mid \langle u, d \rangle \leq \sqrt{\varepsilon}\|d\|, \forall d \in T(S, z)\}. \tag{2.2}$$

When $\varepsilon = 0$, $N_\varepsilon^Q(S, z)$ is the normal cone in the Clarke sense⁶. If S is a convex set and $\varepsilon = 0$, then the set $N_\varepsilon^Q(S, z)$ becomes $N(S, z)$ in convex analysis.

We remark that the function $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is called ε -pseudoconvex at z if

$$g^c(z; d) + \sqrt{\varepsilon}\|d\| \geq 0 \Rightarrow g(z + d) + \sqrt{\varepsilon}\|d\| \geq g(z), \forall d \in \mathbb{R}^n.$$

If g is ε -pseudoconvex at all $z \in \mathbb{R}^n$, then g is called ε -pseudoconvex¹³.

Definition 2.1 Given a locally Lipschitz function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, it is called ε -quasi-convex at z if

$$g(z + d) + \sqrt{\varepsilon}\|d\| \leq g(z) \Rightarrow g^c(z; d) + \sqrt{\varepsilon}\|d\| \leq 0.$$

The function g is called ε -quasi-convex if it is ε -quasi-convex at every $z \in \mathbb{R}^n$.

In preparation for establishing the approximate optimality conditions in Section 3, we recall the notion of approximate quasi-subdifferentials¹³. Given $\varepsilon > 0$, the ε -quasi-subdifferential of a locally Lipschitz function g at z is

$$\partial_\varepsilon^c g(z) := \{u \in \mathbb{R}^n \mid g^c(z; d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d \rangle, \forall d \in \mathbb{R}^n\}. \tag{2.3}$$

The following results are given in¹³:

$$\partial_\varepsilon^c g(z) = \partial^c g(z) + \sqrt{\varepsilon}B, \tag{2.4}$$

$$\partial_\varepsilon^c(\lambda g)(z) = \lambda \partial_\varepsilon^c g(z), \forall \lambda \neq 0. \tag{2.5}$$

For locally Lipschitz functions $g, h: \mathbb{R}^n \rightarrow \mathbb{R}$, we also have

$$\partial_\varepsilon^c(g + h)(z) \subseteq \partial_{\varepsilon_1}^c g(z) + \partial_{\varepsilon_2}^c h(z), \tag{2.6}$$

for all $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$.

Let $h_j: \mathbb{R}^n \rightarrow \mathbb{R}, j \in J := \{1, 2, \dots, m\}$ be locally Lipschitz functions at z . Let $h(z) := \max_{j \in J} h_j(z)$ and $J(z) := \{j \in J \mid h(z) = h_j(z)\}$. Then, for $\varepsilon > 0$, one has

$$\partial_\varepsilon^c h(z) \subseteq \text{co} \left\{ \bigcup_{j \in J(z)} \partial_{\varepsilon_j}^c h_j(z) \right\},$$

where $\sqrt{\varepsilon}$ is a convex combination of $\sqrt{\varepsilon_j}, j \in J(z)$. The equality holds if the functions $h_j, j \in J$ are regular at z ¹³.

For an arbitrary index set Ω , let $\mathbb{R}^\Omega := \{\lambda = (\lambda_\nu) \mid \nu \in \Omega\}$. Consider a linear space $\mathbb{R}^{(\Omega)} :=$

$\{\lambda = (\lambda_v)_{v \in \Omega} \mid \lambda_v = 0, \forall v \in \Omega, \text{ and only finitely many } \lambda_v \neq 0\}$. Then,

$$\mathbb{R}_+^{(\Omega)} = \{\lambda = (\lambda_v)_{v \in \Omega} \in \mathbb{R}^{(\Omega)} \mid \lambda_v \geq 0, v \in \Omega\}$$

is a nonnegative cone of $\mathbb{R}^{(\Omega)}$. For $\lambda \in \mathbb{R}_+^{(\Omega)}$, the supporting set of λ is $\Omega(\lambda) := \{v \in \Omega \mid \lambda_v \neq 0\}^{2,18,19}$. It is a finite subset of Ω .

The following notation, proposed in², will be used in this paper:

$$\|\lambda\|_1 := \sum_{v \in \Omega(\lambda)} |\lambda_v|.$$

We use the notation $\langle \cdot, \cdot \rangle$ to denote a bilinear form on $\mathbb{R}^{(\Omega)} \times \mathbb{R}^{(\Omega)}$. With $\lambda \in \mathbb{R}^{(\Omega)}$ and $z = \{z_v\}_{v \in \Omega} \subset \mathbb{R}^{(\Omega)}$,

$$\langle \lambda, z \rangle := \sum_{v \in \Omega} \lambda_v z_v = \begin{cases} \sum_{v \in \Omega(\lambda)} \lambda_v z_v, & \text{if } \Omega(\lambda) \neq \emptyset \\ 0, & \text{if } \Omega(\lambda) = \emptyset. \end{cases}$$

Some additional notation used in the paper is as follows:

$$\mathbb{R}_+^m := \{(x_1, \dots, x_m) \mid x_i \geq 0, i = 1, \dots, m\},$$

$$\mathbb{R}_{++}^m := \{(x_1, \dots, x_m) \mid x_i > 0, i = 1, \dots, m\}.$$

Consider the following problem:

$$(P) \quad \begin{aligned} &\text{Min} && g(x) \\ &\text{s.t.} && h_v(x) \leq 0, v \in \Omega, \\ &&& x \in S, \end{aligned}$$

where Ω is a compact set, $S \subset \mathbb{R}^n$ is a closed nonempty subset, g is a locally Lipschitz function defined on \mathbb{R}^n , and the constraint functions $h_v, v \in \Omega$, defined on \mathbb{R}^n , are locally Lipschitz functions and uniformly in v .

Let $A := \{x \in S \mid h_v(x) \leq 0, v \in \Omega\}$ be the feasible set of (P). We need the following condition

$$(H): \exists d \in \mathbb{R}^n: h_v^c(z; d) < 0 \text{ and for all } v \in I(z). \tag{2.7}$$

For $\alpha > 0$, we define the relaxation set of A with respect to α by

$$A_\alpha := \{x \in S \mid h_v(x) \leq \sqrt{\alpha}, \forall v \in \Omega\}.$$

Definition 2.2¹³ Let $\alpha > 0$. A point $z \in A_\alpha$ is called

- i) an almost solution up to α if $g(z) \leq g(x) + \alpha$, for all $x \in A$;
- ii) an almost quasi-solution up to α if $g(z) \leq g(x) + \sqrt{\alpha}\|x - z\|$, for all $x \in A$;
- iii) an almost regular solution up to α if z is an almost solution and an almost quasi-solution up to α .

If $z \in A$, we have the corresponding notions of solution, quasi-solution, and regular solution up to α .

We recall the following theorems related to (P). Let $I(z) := \{v \in \Omega \mid h_v(z) = 0\}$. Approximate optimality conditions related to quasi-solutions and almost regular solutions up to α for (P) were given in². Recently, these were restated using approximate quasi-subdifferentials for locally Lipschitz functions. The resulting Corollary 3.6 and Theorem 3.2 from¹⁴ are given below.

Theorem 2.1 Consider the problem (P) and $z \in A$. Suppose that the condition (H) is satisfied. Given $\varepsilon \geq 0$, if z is a quasi-solution up to ε for (P), then there exist $\bar{\varepsilon} \geq 0, \hat{\varepsilon} \geq 0, \bar{\lambda}_v \geq 0, \varepsilon_v \geq 0$, and $v \in \Omega$ such that

$$\begin{cases} 0 \in \partial_{\bar{\varepsilon}}^c g(z) + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \partial_{\varepsilon_v / \bar{\lambda}_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) = 0, v \in \Omega(\bar{\lambda}) \\ \sqrt{\varepsilon} = \sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega(\bar{\lambda})} \sqrt{\varepsilon_v} + \sqrt{\hat{\varepsilon}}. \end{cases}$$

By setting $\bar{\varepsilon}_v = \varepsilon_v / \bar{\lambda}_v^2$, this can be rewritten as follows:

$$\begin{cases} 0 \in \partial_{\bar{\varepsilon}}^c g(z) + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \partial_{\bar{\varepsilon}_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) = 0, v \in \Omega(\bar{\lambda}) \\ \sqrt{\varepsilon} = \sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \sqrt{\bar{\varepsilon}_v} + \sqrt{\hat{\varepsilon}}. \end{cases} \tag{2.8}$$

Theorem 2.2 Let $\varepsilon > 0$. There exists a $z_\varepsilon \in A_\varepsilon$ that is an almost regular solution up to ε for (P), $\bar{\lambda} \in \mathbb{R}_+^{(\Omega)}$ and $\hat{\varepsilon}, \bar{\varepsilon}, \bar{\varepsilon}_v \geq 0$, for all $v \in \Omega$, such that

$$\begin{cases} 0 \in \partial_{\bar{\varepsilon}}^c g(z_\varepsilon) + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \partial_{\bar{\varepsilon}_v}^c h_v(z_\varepsilon) + N_{\bar{\varepsilon}}^Q(S, z_\varepsilon), h_v(z_\varepsilon) > 0, v \in \Omega(\bar{\lambda}) \\ \sqrt{\varepsilon} = \sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \sqrt{\bar{\varepsilon}_v} + \sqrt{\hat{\varepsilon}}. \end{cases}$$

APPROXIMATE SOLUTIONS FOR MULTI-OBJECTIVE PROBLEMS

Consider the following multi-objective problem:

$$(MP) \quad \begin{aligned} &\text{Min} && g(x) := (g_1(x), g_2(x), \dots, g_m(x)) \\ &\text{s.t.} && h_v(x) \leq 0, v \in \Omega, \\ &&& x \in S, \end{aligned}$$

where Ω, S, h_v , and $v \in \Omega$, are as above and $g_i, i \in M := \{1, 2, \dots, m\}$ are locally Lipschitz functions on \mathbb{R}^n . We also denote by A the feasible set of (MP).

Given $\varepsilon := (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{R}_{++}^m$, and $\sqrt{\theta} = \min_{i \in M} \sqrt{\varepsilon_i}$, the relaxation of the feasible set of (MP) is A_θ defined by

$$A_\theta := \{x \in S \mid h_\nu(x) \leq \sqrt{\theta}, \nu \in \Omega\}.$$

The following notation is from Definition 2.9 and Definition 2.10 in 2¹.

Definition 3.1 A point $z \in A_\theta$ is said to be

- i) an almost Pareto solution up to ε for (MP) if there exists no $x \in A$ such that

$$\begin{cases} g_i(x) \leq g_i(z) - \varepsilon_i, \forall i \in M, \\ \exists j \in M: g_j(x) < g_j(z) - \varepsilon_j; \end{cases}$$
- ii) an almost quasi-Pareto solution up to ε for (MP) if there exist no $x \in A$ such that

$$\begin{cases} g_i(x) + \sqrt{\varepsilon_i} \|x - z\| \leq g_i(z), \forall i \in M, \\ \exists j \in M: g_j(x) + \sqrt{\varepsilon_j} \|x - z\| < g_j(z); \end{cases}$$
- iii) an almost regular Pareto solution up to ε for (MP) if z is an almost Pareto solution and an almost quasi-Pareto solution up to ε for (MP).

Definition 3.2 A point $z \in A_\theta$ is said to be

- i) an almost weakly Pareto solution up to ε for (MP) if there exists no $x \in A$ such that

$$g_i(x) < g_i(z) - \varepsilon_i, \forall i \in M;$$
- ii) an almost weakly quasi-Pareto solution up to ε for (MP) if there exists no $x \in A$ such that

$$g_i(x) + \sqrt{\varepsilon_i} \|x - z\| < g_i(z), \forall i \in M;$$
- iii) an almost weakly regular Pareto solution up to ε for (MP) if it is an almost weakly Pareto solution and an almost weakly quasi-Pareto solution up to (MP).

In Definitions 3.1 and 3.2, if $z \in A$, then the word “almost” is removed.

Several scalarization methods are used to study Pareto solutions of problem (MP). In this study, we focus on the ε -constraint scalarization method^{11,12}. Consider the general multi-objective optimization problem

$$\begin{aligned} \text{Min} \quad & (g_1(x), \dots, g_m(x)) \\ & x \in S, \end{aligned} \tag{3.1}$$

where $g_i: \mathbb{R}^n \rightarrow \mathbb{R}$ for $i \in M := \{1, 2, \dots, m\}$, and $S \subset \mathbb{R}^n$. When using this method, the scalar optimization problem associated with (3.1) corresponding to the j th objective is given by

$$\begin{aligned} \text{Min} \quad & g_j(x) \\ \text{s.t.} \quad & g_i(x) \leq \varepsilon_i, \quad i \in M \setminus \{j\}, \\ & x \in S, \end{aligned} \tag{3.2}$$

where $\varepsilon \in \mathbb{R}^m$. Here, the components ε_i are prescribed upper bounds, and the constraints $g_i(x) \leq \varepsilon_i$ may or may not be active at the optimal solution of the problem. The values ε_i can be selected to ensure feasibility and to reflect the preferences of the decision maker. By varying them, different Pareto solutions of the problem (3.1) can be found according to their relationships with optimal solutions of (3.2). In other words, the feasible set of (3.2) is properly modified to ensure that Pareto solutions of the problem (3.1) can be obtained^{11,12}. Typically, ε_i can be replaced by $g_i(z)$, where z is a candidate approximate Pareto solution of the problem (3.1). Motivated by this approach, we propose the following modification to study approximate Pareto solutions of the problem (MP).

Given $\varepsilon \in \mathbb{R}_{++}^m$ and $z \in S$, consider the scalar problem associated with problem (MP):

$$\begin{aligned} (P_j(z)) \quad \text{Min} \quad & g_j(x) \\ \text{s.t.} \quad & g_i(x) + \sqrt{\varepsilon_i} \|x - z\| \leq g_i(z), \forall i \in M \setminus \{j\}, \\ & h_\nu(x) \leq 0, \nu \in \Omega, \\ & x \in S. \end{aligned}$$

Let $A_j(z)$ be the set of feasible points of the problem $(P_j(z))$ and assume that $A_j(z) \neq \emptyset$.

Lemma 3.1 Given $\varepsilon \in \mathbb{R}_{++}^m$, set $\sqrt{\theta} := \min_{i \in M} \sqrt{\varepsilon_i}$. The point $z \in A_\theta$ is an almost quasi-solution up to ε_j for the problem $(P_j(z))$ with each $j \in M$ if and only if z is an almost quasi-Pareto solution up to ε for (MP).

Proof. (\Rightarrow) Suppose that z is an almost quasi-solution up to ε_j of the problem $(P_j(z))$ for each $j \in M$ and that z is not an almost quasi-Pareto solution up to ε of problem (MP). By Definition 3.1, there exists $x' \in A$ such that

$$\begin{cases} g_i(x') + \sqrt{\varepsilon_i} \|x' - z\| \leq g_i(z), \forall i \in M, \\ \exists j \in M: g_j(x') + \sqrt{\varepsilon_j} \|x' - z\| < g_j(z). \end{cases}$$

This result contradicts the assumption that z is an almost quasi-solution up to ε_j of the problem $(P_j(z))$ for each $j \in M$.

(\Leftarrow) Suppose that z is an almost quasi-Pareto solution up to ε of problem (MP) and that z is not an almost quasi-solution up to ε_j of any problem $(P_j(z)), j \in M$. Then, there exist $j' \in M$ and $x' \in A_{j'}(z)$ such that

$$g_{j'}(x') + \sqrt{\varepsilon_{j'}} \|x' - z\| < g_{j'}(z).$$

Hence, we reach a contradiction with the assumption that z is an almost quasi-Pareto solution up to ε for the problem (MP). ■

Lemma 3.2. If there exists $j \in M$ such that z is an almost quasi-solution up to ε_j of the problem $(P_j(z))$ then

z is an almost weakly quasi-Pareto solution up to ε of the problem (MP).

Proof. Suppose there exists $j \in M$ such that z is an almost quasi-solution up to ε_j of the problem $(P_j(z))$ and z is not an almost weakly quasi-Pareto solution up to ε of the problem (MP). Then, there exists $x' \in A$ such that

$$g_i(x') + \sqrt{\varepsilon_i} \|x' - z\| < g_i(z), \forall i \in M.$$

Hence, there exists $x' \in A$ for which the above inequality holds for j . This is contrary to the assumption that z is an almost quasi-solution up to ε_j for $(P_j(z))$.

Remark 3.1. In Lemma 3.1, if we take $z \in A$, then the word ‘‘almost’’ can be omitted.

Analogous to Lemma 3.1, the following lemma holds for weak solutions.

Lemma 3.3. Given $\varepsilon \in \mathbb{R}_{++}^m$, suppose that $\sqrt{\varepsilon_j} := \min_{i \in M} \sqrt{\varepsilon_i}$. Then, $z \in A_{\varepsilon_j}$ is an almost weakly quasi-Pareto solution up to ε_j of the problem $(P_j(z))$ if and only if z is an almost weakly quasi-Pareto solution up to ε of the problem (MP).

Approximate optimality conditions for $(P_j(z))$

Let $x \in S, I(x) = \{v \in \Omega \mid h_v(x) = 0\}$, and

$$H_j(x) = \{i \in M \setminus \{j\} \mid g_i(x) + \sqrt{\varepsilon_i} \|x - z\| = g_i(z)\}.$$

We need to define the following condition:

$$(\mathcal{A}_j): \exists d \in T(S, x): \begin{cases} h_v^c(x; d) < 0 & \text{for all } v \in I(x), \\ g_i^c(x; d) + \sqrt{\varepsilon_i} \|d\| < 0 & \text{for all } i \in H_j(x). \end{cases}$$

Theorem 3.1. For $\varepsilon \in \mathbb{R}_{++}^m$, suppose that $z \in A_j(z)$ is a quasi-solution up to ε_j of $(P_j(z))$ and the condition (\mathcal{A}_j) is satisfied for z . Then, there exist $\mu_i \geq 0, i \in M \setminus \{j\}, \lambda \in \mathbb{R}_+^{(\Omega)}$ and $\bar{\varepsilon}, \varepsilon'_i, \varepsilon'_v, \varepsilon' \geq 0$, with $v \in \Omega(\lambda)$, such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\varepsilon'}^Q(S, z), h_v(z) = 0, \forall v \in \Omega(\lambda), \quad (3.3)$$

with

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega(\lambda)} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}. \quad (3.4)$$

Proof. Consider the function l_k given by

$$l_k(x) = \begin{cases} g_i(x) + \sqrt{\varepsilon_i} \|x - z\| - g_i(z), k \in M \setminus \{j\}, \\ h_v(x), k \in \Omega. \end{cases}$$

Since $z \in A_j(z)$ is a quasi-solution up to ε_j of problem $(P_j(z))$ and condition (\mathcal{A}_j) is satisfied, by applying Theorem 2.1 to problem $(P_j(z))$ with the feasible set

$$A_j(z) = \{x \in S \mid l_k(x) \leq 0, k \in K := \Omega \cup M \setminus \{j\}\}, \quad (3.5)$$

it can be seen that there exist $\bar{\varepsilon}, \varepsilon'_k, \bar{\varepsilon}', \lambda_k \geq 0$ such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{k \in K} \lambda_k \partial_{\varepsilon'_k}^c l_k(z) + N_{\bar{\varepsilon}'}^Q(S, z), \quad (3.6)$$

with

$$\sqrt{\bar{\varepsilon}} + \sum_{k \in K} \lambda_k \sqrt{\varepsilon'_k} + \sqrt{\bar{\varepsilon}'} = \sqrt{\varepsilon_j}, \lambda_k l_k(z) = 0, \forall k \in K$$

Since $\partial_{\varepsilon'_i}^c l_i(z) \subseteq \partial_{\varepsilon'_i}^c g_i(z) + \sqrt{\varepsilon'_i} B$, with $i \in M \setminus \{j\}$, and B is a closed unit ball in \mathbb{R}^n , the inclusion (3.6) can be rewritten with $\mu_i, i \in M \setminus \{j\}$ replacing $\lambda_k, k \in M \setminus \{j\}$ as follows:

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}'}^Q(S, z) + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} B \quad (3.7)$$

and $\lambda_k l_k(z) = 0, \forall k \in \Omega \cup M \setminus \{j\}$. Because of the structure of the functions l_k and the set $\Omega(\lambda)$, the complementary condition becomes $h_v(z) = 0, v \in \Omega(\lambda)$. Furthermore, we have

$$N_{\bar{\varepsilon}'}^Q(S, z) + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} B = N(S, z) + \sqrt{\bar{\varepsilon}'} B + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} B.$$

Let $\sqrt{\varepsilon'} = \sqrt{\bar{\varepsilon}'} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i}$. From (3.7), we deduce that

$$0 \in \partial_{\varepsilon'}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\varepsilon'}^Q(S, z), h_v(z) = 0, \forall v \in \Omega(\lambda)$$

with

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}$$

Similar to Theorem 2.2, the existence of an approximate solution for the problem is demonstrated as follows.

Theorem 3.2. Let $\varepsilon \in \mathbb{R}_{++}^m$. For each $j \in M$, there exists a z that is an almost regular solution up to ε_j for $(P_j(z))$, $\mu_i \geq 0, i \in M \setminus \{j\}, \lambda \in \mathbb{R}_+^{(\Omega)}$, and $\bar{\varepsilon}, \varepsilon'_i, \varepsilon'_v, \varepsilon' \geq 0$, with $v \in \Omega(\lambda)$, such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\varepsilon'}^Q(S, z), h_v(z) > 0, \forall v \in \Omega(\lambda), \quad (3.8)$$

where

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}. \tag{3.9}$$

Proof. Let $A_j^{\varepsilon_j}(z)$ be a relaxation of the set $A_j(z)$ with $\sqrt{\varepsilon_j}$. By applying Theorem 2.2, it can be shown that there exists a $z \in A_j^{\varepsilon_j}(z)$ that is an almost quasi-solution up to ε_j for $(P_j(z))$ and $\bar{\varepsilon}, \varepsilon'_k, \bar{\varepsilon}', \lambda_k \geq 0$ such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{k \in K} \lambda_k \partial_{\varepsilon'_k}^c l_k(z) + N_{\bar{\varepsilon}'}^Q(S, z), \tag{3.10}$$

where $K := \Omega \cup M \setminus \{j\}$ and

$$\sqrt{\bar{\varepsilon}} + \sum_{k \in K} \lambda_k \sqrt{\varepsilon'_k} + \sqrt{\bar{\varepsilon}'} = \sqrt{\varepsilon_j}, h_k(z) > 0, \forall k \in K(\lambda).$$

Note that $\Omega \cap M \setminus \{j\} = \emptyset$, so $K(\lambda) := [(M \setminus \{j\}) \cup \Omega](\lambda) = (M \setminus \{j\})(\lambda) \cup \Omega(\lambda)$. From the structure of $l_k(z)$, the conditions $l_k(z) > 0$ for all $k \in K(\lambda)$ reduce to $h_v(z) > 0$ for all $v \in \Omega(\lambda)$. The result follows by arguing as in the proof of Theorem 3.1.

■

Analogous to the argument in [20, Proposition 3.2], we derive the following sufficient condition by additionally assuming that S is a closed convex set.

Theorem 3.3. *For the problem $(P_j(z))$, let $\varepsilon_j > 0$ and $z \in A_j(z)$. Suppose that g_j is an ε_j -pseudoconvex function and that g_i and h_v are ε_i -quasi-convex and quasi-convex functions with $i \in M \setminus \{j\}$ and $v \in \Omega$, respectively. If there exist $\mu_i \geq 0, i \in M \setminus \{j\}, \lambda \in \mathbb{R}_+^{(\Omega)}$, and $\bar{\varepsilon}, \varepsilon'_i, \bar{\varepsilon}', \varepsilon'_v \geq 0$ for all $v \in \Omega$ such that the conditions (3.3) and (3.4) are satisfied, then z is a quasi-solution up to ε_j for the problem $(P_j(z))$.*

Proof. From conditions (3.3) and (3.4), there exist $u_j \in \partial^c g_j(z), u_i \in \partial^c g_i(z), i \in M \setminus \{j\}, v_v \in \partial^c h_v(z), v \in \Omega(\lambda)$, and $w \in N(S, z)$ such that

$$0 = u_j + \sqrt{\bar{\varepsilon}}B + \sum_{i \in M \setminus \{j\}} \mu_i (u_i + \sqrt{\varepsilon'_i})B + \sum_{v \in \Omega(\lambda)} \lambda_v (v_v + \sqrt{\varepsilon'_v})B + w + \sqrt{\bar{\varepsilon}'}B,$$

that is,

$$u_j + \left(\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega(\lambda)} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} \right) B = - \sum_{i \in M \setminus \{j\}} \mu_i u_i - \sum_{v \in \Omega(\lambda)} \lambda_v v_v - w, \tag{3.11}$$

with

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}, h_v(z) = 0, \forall v \in \Omega(\lambda).$$

Since $g_i, i \in M \setminus \{j\}$, and $h_v, v \in \Omega$, are ε_i -quasi-convex and quasi-convex functions, respectively, $g_i(x) + \sqrt{\varepsilon_i} \|x - z\| \leq g_i(z)$ with $i \in M \setminus \{j\}, x \in A_j(z)$, and $h_v(x) \leq h_v(z)$, for $x \in A_j(z)$. Thus, we can deduce that $\langle u_i, x - z \rangle \leq -\sqrt{\varepsilon_i} \|x - z\|$ with $i \in M \setminus \{j\}$ and $\langle v_v, x - z \rangle \leq 0$ with $v \in \Omega$. We also have $\langle w, x - z \rangle \leq 0$ for all $x \in N(S, z)$, so $\langle w, x - z \rangle \leq 0$ for all $x \in A_j(z)$. Hence, $\langle u_j, x - z \rangle + \sqrt{\varepsilon_j} \|x - z\| \geq 0$. By the ε_j -pseudoconvexity of g_j , we obtain $g_j(x) + \sqrt{\varepsilon_j} \|x - z\| \geq g_j(z)$ for all $x \in A_j(z)$, that is, z is a quasi-solution up to ε_j for the problem $(P_j(z))$.

■

Approximate optimality conditions for (MP)

Lemma 3.4. *One has $\lambda N_{\bar{\varepsilon}}^Q(K, z) = N_{\lambda^2 \bar{\varepsilon}}^Q(K, z), \forall \lambda > 0$.*

Proof. With $\lambda > 0$, we have

$$\begin{aligned} \lambda N_{\bar{\varepsilon}}^Q(K, z) &= \lambda \{u \in \mathbb{R}^n \mid \langle u, d \rangle \leq \sqrt{\bar{\varepsilon}} \|d\|, \forall d \in T(K, z)\} \\ &= \{\lambda u \in \mathbb{R}^n \mid \langle \lambda u, d \rangle \leq \lambda \sqrt{\bar{\varepsilon}} \|d\|, \forall d \in T(K, z)\} \\ &= N_{\lambda^2 \bar{\varepsilon}}^Q(K, z). \end{aligned}$$

■

Theorem 3.4. *Let $\varepsilon \in \mathbb{R}_+^m$ and let $z \in A$ be a quasi-Pareto solution up to ε of (MP). Suppose there exists $j \in M$ such that (\mathcal{A}_j) is satisfied for z . Then, there exist $\bar{\mu}_i \geq 0, i \in M, \sum_{i \in M} \bar{\mu}_i = 1, \varepsilon'_i, \varepsilon'_v, \bar{\varepsilon}' \geq 0$, and $\bar{\lambda} \in \mathbb{R}_+^{(\Omega)}$ such that*

$$0 \in \sum_{i \in M} \bar{\mu}_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\bar{\lambda})} \bar{\lambda}_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}'}^Q(S, z), h_v(z) = 0, \forall v \in \Omega(\bar{\lambda}), \tag{3.12}$$

with

$$\sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \bar{\lambda}_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon_i}. \tag{3.13}$$

Proof. By Lemma 3.1 and Remark 3.1, if z is a quasi-Pareto solution up to ε of (MP), then z is a quasi-solution up to ε_j for problem $(P_j(z))$. Since the condition (\mathcal{A}_j) is satisfied for z , by Theorem 3.1, there exist $\mu_i \geq 0, i \in M \setminus \{j\}, \lambda \in \mathbb{R}_+^{(\Omega)}$, and $\bar{\varepsilon}, \varepsilon'_i, \varepsilon'_v, \bar{\varepsilon}' \geq 0$ such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}'}^Q(S, z), h_v(z) = 0, \forall v \in \Omega(\lambda), \tag{3.14}$$

with

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}. \tag{3.15}$$

Multiplying both the sides of the equality (3.14) by $\frac{1}{1 + \sum_{i \in M \setminus \{j\}} \mu_i}$ leads to

$$0 \in \frac{1}{1 + \sum_{i \in M \setminus \{j\}} \mu_i} \left(\partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z) \right).$$

If we set

$$\begin{aligned} \bar{\mu}_j &= \frac{1}{1 + \sum_{i \in M \setminus \{j\}} \mu_i}, \bar{\mu}_i = \frac{\mu_i}{1 + \sum_{i \in M \setminus \{j\}} \mu_i}, i \in M \setminus \{j\} \\ \bar{\lambda}_v &= \frac{\lambda_v}{1 + \sum_{i \in M \setminus \{j\}} \mu_i}, v \in \Omega(\lambda), \sqrt{\bar{\varepsilon}'} = \frac{\sqrt{\varepsilon'}}{1 + \sum_{i \in M \setminus \{j\}} \mu_i} \\ \sqrt{\bar{\varepsilon}} &= \sqrt{\varepsilon'_j}, \end{aligned}$$

then, from Lemma 3.4, we obtain

$$0 \in \sum_{i \in M} \bar{\mu}_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega} \bar{\lambda}_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z).$$

From (3.15), we deduce that

$$\sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \bar{\lambda}_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon_i}, h_v(z) = 0, \forall v \in \Omega(\bar{\lambda}).$$

We can check that $\sum_{i \in M} \bar{\mu}_i = 1$.

By applying Theorem 2.2 and Lemma 3.1, we deduce the existence of an almost Pareto approximate solution for problem (MP) through the following theorem.

Theorem 3.5. For $\varepsilon \in \mathbb{R}_{++}^m$, suppose that $\sqrt{\varepsilon_j} := \min_{i \in M} \sqrt{\varepsilon_i}$. Then, there exists a $z \in A_{\varepsilon_j}$ that is an almost regular Pareto solution up to ε of the problem (MP), together with $\bar{\mu}_i \geq 0, i \in M, \sum_{i \in M} \bar{\mu}_i = 1, \varepsilon'_i, \varepsilon'_v, \varepsilon' \geq 0$, and $\bar{\lambda} \in \mathbb{R}_+^{(\Omega)}$, such that

$$0 \in \sum_{i \in M} \bar{\mu}_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \bar{\lambda}_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) > 0, \forall v \in \Omega(\bar{\lambda}), \tag{3.16}$$

where

$$\sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \bar{\lambda}_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon_i}. \tag{3.17}$$

Proof. By Theorem 3.2, there exists a $z \in A_{\varepsilon_j}$ that is an almost regular solution up to ε of $(P_j(z))$, together with $\mu_i \geq 0, i \in M \setminus \{j\}, \lambda \in \mathbb{R}_+^{(\Omega)}, \bar{\varepsilon}, \varepsilon'_i, \varepsilon'_v, \varepsilon' \geq 0$, and $v \in \Omega(\lambda)$ such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{i \in M \setminus \{j\}} \mu_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) > 0, \forall v \in \Omega(\lambda), \tag{3.18}$$

where

$$\sqrt{\bar{\varepsilon}} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j} + \sum_{i \in M \setminus \{j\}} \mu_i \sqrt{\varepsilon_i}. \tag{3.19}$$

By an argument analogous to the proof of Theorem 3.4, (3.18) and (3.19) can be rewritten as

$$0 \in \sum_{i \in M} \bar{\mu}_i \partial_{\varepsilon'_i}^c g_i(z) + \sum_{v \in \Omega(\lambda)} \bar{\lambda}_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) > 0, \forall v \in \Omega(\bar{\lambda}),$$

where

$$\sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon'_i} + \sum_{v \in \Omega} \bar{\lambda}_v \sqrt{\varepsilon'_v} + \sqrt{\bar{\varepsilon}'} = \sum_{i \in M} \bar{\mu}_i \sqrt{\varepsilon_i}.$$

By Lemma 3.1, z is an almost regular Pareto solution up to ε for the problem (MP).

Theorem 3.6. Suppose we are given $\varepsilon \in \mathbb{R}_{++}^m, \sqrt{\varepsilon_j} := \min_{i \in M} \sqrt{\varepsilon_i}$, and $z \in A_{\varepsilon_j}$. For (MP), assume that S is a closed convex set. Suppose that there exists $j \in M$ such that g_j is an ε_j -pseudoconvex function and that $h_v, v \in \Omega(\lambda)$ are quasi-convex functions. If there exists $\lambda \in \mathbb{R}_+^{(\Omega)}$ and $\bar{\varepsilon}, \varepsilon'_v, \varepsilon' \geq 0$ such that

$$0 \in \partial_{\bar{\varepsilon}}^c g_j(z) + \sum_{v \in \Omega(\lambda)} \lambda_v \partial_{\varepsilon'_v}^c h_v(z) + N_{\bar{\varepsilon}}^Q(S, z), h_v(z) > 0, \forall v \in \Omega(\lambda), \tag{3.20}$$

where

$$\sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j},$$

then z is an almost weakly Pareto solution up to ε of (MP).

Proof. By the assumption $h_v(z) > 0, \forall v \in \Omega(\lambda)$, we have $z \notin A$. From (3.20), there exist $u \in \partial^c g_j(z), v'_v \in \partial^c h_v(z), v \in \Omega(\lambda)$, and $w \in N(S, z)$ such that

$$(u + \sqrt{\bar{\varepsilon}}B) + \sum_{v \in \Omega(\lambda)} \lambda_v (v'_v + \sqrt{\varepsilon'_v})B + (w + \sqrt{\varepsilon'}B) = 0,$$

where

$$\sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'} = \sqrt{\varepsilon_j}.$$

This implies that

$$u + (\sqrt{\bar{\varepsilon}} + \sum_{v \in \Omega(\lambda)} \lambda_v \sqrt{\varepsilon'_v} + \sqrt{\varepsilon'})B = -\sum_{v \in \Omega(\lambda)} \lambda_v v'_v - w,$$

that is,

$$u + \sqrt{\varepsilon_j}B = -\sum_{v \in \Omega(\lambda)} \lambda_v v'_v - w.$$

Hence,

$$\langle u, x - z \rangle + \sqrt{\varepsilon_j} \|x - z\| \geq -\sum_{v \in \Omega(\lambda)} \lambda_v \langle v, x - z \rangle - \langle w, x - z \rangle.$$

Observe that the right-hand side of the above inequality is nonnegative because the functions $h_v, v \in \Omega(\lambda)$ are quasi-convex and S is a convex set. Therefore,

$$\langle u, x - z \rangle + \sqrt{\varepsilon_j} \|x - z\| \geq 0.$$

Since g_j is ε_j -pseudoconvex,

$$g_j(x) + \sqrt{\varepsilon_j} \|x - z\| \geq g_j(z). \tag{3.21}$$

To prove that z is an almost weakly Pareto solution up to ε for (MP), we have to show that $h_v(z) \leq \sqrt{\varepsilon_j}$. Note that $z \in A_{\varepsilon_j}$ and $g_v(z) > 0$, for all $v \in \Omega(\lambda)$. From (3.21), by applying Lemma 3.2, the desired result follows.

Example 3.1. The following example provides an illustration of Theorem 3.5:

$$\begin{aligned} \text{(MP)} \quad & \text{Min } (|x + y|, |x - y|) \\ \text{s.t.} \quad & x + y - \frac{\sqrt{3} - 1}{2} \geq 0, \\ & y \geq 0. \end{aligned}$$

Let $g_1(x, y) = |x + y|$, $g_2(x, y) = |x - y|$, $h(x, y) = \frac{\sqrt{3}-1}{2} - x - y$, and $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Given $\varepsilon = (\varepsilon_1, \varepsilon_2) = (\frac{3}{4}, \frac{3}{4})$, we have $\sqrt{\theta} = \min \sqrt{\varepsilon_i} = \frac{\sqrt{3}}{2}, \forall i = 1, 2$. The relaxation of the feasible set A of (MP) with $\sqrt{\theta}$ is

$$A_{\theta} = \{(x, y) \mid x + y + \frac{1}{2} \geq 0, y \geq 0\}.$$

We will show that $(0, 0)$ is an almost quasi-weakly Pareto solution up to ε for the problem (MP) that satisfies Theorem 3.5. It can be seen that $(0, 0) \notin A$, but $(0, 0) \in A_{\theta}$. Moreover, we can check that the point $z = (0, 0)$ is an almost weakly quasi-Pareto solution up to ε for (MP) according to Definition 3.2. Next, we demonstrate that the approximate optimality condition in Theorem 3.5 holds. A simple computation shows that

$$\begin{aligned} \partial^c g_1(0, 0) &= \text{co}\{(-1, -1), (1, 1)\}, \\ \partial^c g_2(0, 0) &= \text{co}\{(1, -1), (-1, 1)\}, \\ \partial^c h(0, 0) &= \{(-1, -1)\}, \\ N(S, (0, 0)) &= \{(u, v) \mid u = 0, v \leq 0\}. \end{aligned}$$

Let $\varepsilon^* = \varepsilon = \frac{3}{64}$ and $\hat{\varepsilon} = \frac{3}{16}$. We can check that

$$\begin{aligned} \left(1 - \frac{\sqrt{3}}{16}, 1 - \frac{\sqrt{3}}{16}\right) &= (1, 1) + \frac{\sqrt{3}}{8} \left(-\frac{1}{2}, -\frac{1}{2}\right) \in \partial^c f_1(0, 0) + \sqrt{\varepsilon^*} B = \partial_{\varepsilon^*}^c f_1(0, 0), \\ \left(-1 + \frac{\sqrt{3}}{16}, -1 + \frac{\sqrt{3}}{16}\right) &= (-1, -1) + \frac{\sqrt{3}}{8} \left(\frac{1}{2}, \frac{1}{2}\right) \in \partial^c g(0, 0) + \sqrt{\varepsilon} B = \partial_{\varepsilon}^c g(0, 0), \\ (0, 0) &= \left(0, -\frac{\sqrt{3}}{4}\right) + \left(\frac{\sqrt{3}}{4}\right) (0, 1) \in N(S, (0, 0)) + \sqrt{\hat{\varepsilon}} B = N_{\hat{\varepsilon}}(S, (0, 0)). \end{aligned}$$

If we choose $\mu_1 = 1, \mu_2 = 0, \lambda = 1$, and $\varepsilon'_1 = \varepsilon'_2 = \varepsilon^*$, then $(0, 0) \in \sum_{i=1}^2 \mu_i \partial_{\varepsilon'_i}^c g_i(0, 0) + \lambda \partial_{\hat{\varepsilon}}^c h(0, 0) + N_{\hat{\varepsilon}}(S, (0, 0))$. It can be seen that $0 < h(0, 0) \leq \sqrt{\theta}$ and $\sqrt{\theta} = \sqrt{\varepsilon^*} + \lambda \sqrt{\hat{\varepsilon}} + \sqrt{\hat{\varepsilon}}$. Thus, z is an almost weakly quasi-Pareto solution up to ε for (MP) that satisfies the approximate optimality condition of Theorem 3.5.

CONCLUSIONS

Using the new concepts of ε -quasi-subdifferentials and ε -quasi-normal sets, this paper presents optimality conditions for approximate solutions of a nonconvex multi-objective problem with infinitely many constraints. The results are based on a modified scalarization method for multi-objective optimization problems. They provide an alternative approximate representation of the optimality conditions associated with approximate solutions of nonconvex multi-objective optimization problems, which differs from existing results.

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CONFLICT OF INTEREST

No potential conflict of interest was reported by the authors.

AUTHOR CONTRIBUTIONS

Author Hua Khac Bao conducted the research and prepared the first draft of the manuscript. Author Ta Quang Son supervised the research and contributed to the revision and improvement of the manuscript

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